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The Sasaki Hook is not a [Static] Implicative Connective but Induces a Backward [in Time] Dynamic One that Assigns Causes

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The Sasaki adjunction, which formally encodes the logicality that different authors tried to attach to the Sasaki hook as a 'quantum implicative connective,' has a fundamental dynamic nature and encodes the so-called 'causal duality' (Coecke *et al.*, 2001) for the particular case of a quantum measurement with a projector as corresponding self-adjoint operator. The action of the Sasaki hook $(a \stackrel{S}{\to} -)$ for fixed antecedent a assigns to some property "the weakest cause before the measurement of actuality of that property after the measurement," i.e., $(a \stackrel{S}{\to} b)$ is the weakest property that guarantees actuality of b after performing the measurement represented by the projector that has the 'subspace a' as eigenstates for eigenvalue 1, say, the measurement that 'tests' a. The logicality attributable to quantum systems contains a fundamentally dynamic ingredient: Causal duality actually provides a new dynamic interpretation of orthomodularity. We also reconsider the status of the Sasaki hook within 'dynamic (operational) quantum logic,' what leads us to the claim made in the title of this paper. The Sasaki adjunction has a physical significance in terms of causal duality. The labeled dynamic hooks (forwardly and backwardly) that encode quantum measurements, act on properties as $(a_1 \stackrel{\varphi_a}{\to} a_2) := (a_1 \to_L (a \stackrel{S}{\to} a_2))$ and $(a_1 \stackrel{\varphi_a}{\leftarrow} a_2) := ((a \stackrel{S}{\to} a_2) \to_L a_1)$, taking values in the 'disjunctive extension' $\mathrm{DI}(L)$ of the property lattice L, where $a \in L$ is the tested property and $(- \to_L -)$ is the Heyting implication that lives on $\mathrm{DI}(L)$. Since these hooks $(- \stackrel{\varphi_a}{\to} -)$ and $(- \stackrel{\varphi_a}{\leftarrow} -)$ extend to $\mathrm{DI}(L) \times \mathrm{DI}(L)$ they constitute internal operations. The transition from either classical or constructive/intuitionistic logic to quantum logic entails besides the introduction of an additional unary connective 'operational resolution' (Coecke, 2002a) the shift from a binary connective implication to a ternary connective where two of the arguments refer to qualities of the system and the third, the new one, to an obtained outcome (in a measurement).

KEY WORDS: Sasaki hook; dynamic quantum logic; galois adjoint.

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1. QUANTUM LOGICALITY

We claim that logical considerations on quantum behavior and as such, further development of the research field, have been 'corrupted' by two features. Once these two features are neutralized, the way towards an essentially dynamic quantum logic (i.e., a unified logic of 'changes' both for classical and quantum systems), or otherwise put, a true quantum process semantics, is opened. Moreover, the solution to the second 'corrupt feature' indicates that the logicality encoded in pure quantum theory is of a fundamental dynamic nature. Structures somewhat similar to those emerging in the context of categorical grammar (Lambek, 1958), linear logic (Girard, 1987, 2000), action logic (Baltag, 1999) and computation and concurrency (Abramsky, 1993; Milner, 1999) then naturally emerge via a Kripke style approach for logical semantics applied to the operational foundations of physics. The two features that obstructed true logicality are—concerning the second, most non-quantum logicians have always agreed on its weakness:

i. The Birkhoff and von Neumann (1936) 'dilemma': "Whereas logicians have usually assumed that [the orthocomplementation] properties L71-L73 of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the distributive identities L6 as the weakest link in the algebra of logic." This dilemma forced the search and attempted identification of quantum logicality to proceed 'orthogonal' to intuitionistic and derived developments in logic.
ii. 'Implication' via the Sasaki adjunction: The fact that the pointwise action $\varphi_a^*(-)$ of Hilbert space projectors on the subspace lattice, the quantum analogue of the action of classical lattice projections $(a \wedge -)$, has the parameterized action $(a \stackrel{S}{\to} -)$ of the so-called Sasaki hook $(- \stackrel{S}{\to} -)$ as a right adjoint. This Sasaki hook (as a binary operation) satisfies the minimal implicative condition $(a \stackrel{S}{\to} b) = 1 \Longleftrightarrow a \leq b$ (Kalmbach, 1983) where $\leq$ naturally encodes physical consequence (Coecke *et al.*, 2001a).[4] However, any proof theoretic consideration (among other things) did turn out to be impossible for a logical system with $(- \stackrel{S}{\to} -)$ as implication since there cannot be a deduction theorem for it (Blok *et al.*, 1984; Malinowski 1990).[5] We explain all this in more detail in Section 3.

[4] Recall that the adjointness of projection $(a \wedge -)$ and implication $(a \to -)$ in classical and intuitionistic logic exactly encodes the validity of modus ponens and deduction, in other words, the adjunction (sometimes called the 'implicative condition') $a \wedge x \leq b \Longleftrightarrow x \leq (a \to b)$ is equivalent to $a \wedge x \leq b \Longrightarrow x \leq (a \to b)$ together with $a \wedge (a \to b) \leq b$. By means of applying the latter, i.e., modus ponens, given that $x \leq (a \to b)$ we indeed obtain $a \wedge x \leq a \wedge (a \to b) \leq b$. We come back to this further in this paper.

[5] In Hardegree (1975, 1979) and Herman *et al.* (1975) it is pointed out that the Sasaki hook also fails to satisfy strong transitivity, weakening and contraposition.

It turns out that an operational analysis of quantum logicality starting from well-defined primitive notions rather than from formal pragmatism eliminates these two features. Instead:

i. As shown in Coecke (2002a), the injective hull construction for meet-semilattices (Bruns and Lakser, 1970; Horn and Kimura, 1971) realizes a disjunctive extension of property lattices, the latter being the physical incarnation of meet-complete and conjunctive quantum logicality (and nothing more!),[6] in terms of a complete Heyting algebra that goes equipped with an additional operation, 'operational resolution', which recaptures the initial property lattice as its range, and this goes without any loss of the (physically derivable) logical content of the initial lattice of properties. In the case of an atomistic property lattice, the inclusion of the property lattice in its distributive hull actually encodes the 'state space – property lattice duality' (Coecke, 2002b).[7] This construction will be recalled in the fourth section of this paper.
ii. Propagation of physical properties is left adjoint to backward causal assignment (Coecke *et al.*, 2001)—we provide a more intuitive presentation of this result in the Section 5. Also in Section 5, we show that the Sasaki adjunction exactly encodes this adjunction for the case of propagation and backward causal assignment of a quantum measurement. The minimal implicative condition expresses in this perspective merely that the image under projection of the trivial property 1 is exactly the property on which we project, i.e., $\varphi_a^*(1) = a$. More important however, recalling that adjointness of Sasaki hook and Sasaki projection for an ortholattice is equivalent to the ortholattice being orthomodular, causal duality provides actually a new interpretation of orthomodularity.

Moreover, as shown in Coecke (2002b) and Coecke *et al.* (2001b), when combining the following features that result from the above:

i. Property lattices admit a canonical disjunctive extension giving rise to an irredundant collection of meaningful propositions on properties with a physically significant ordering,[8]
ii. A unary connective 'operational resolution' faithfully recaptures the physical properties within the collection of propositions on these properties as its range, and,

[6] For a demonstration of complete conjunctivity see Piron (1976) and Moore (1999). For the 'and nothing more!' see Emch and Jauch (1965), Coecke (2002a) and Coecke *et al.* (2001a).

[7] For a discussion of the categorical 'state space–property lattice duality' for atomistic orthocomplemented lattices, physically and mathematically, see Moore (1995).

[8] For a clear distinction between the significance of 'properties' and 'propositions on properties,' we initially refer to Coecke (2002a) and the rest of this paper. Briefly, from a philosophical perspective one could say that properties are ontological, there where propositions on properties that are to be situated at an epistemological meta-level.

iii. causal duality applies both to properties and to propositions on properties, respectively restricting physically admissible evolution, and encoding preservation of propositional disjunction.

Then, a Kripke-style approach for logical semantics applied to the operational foundations of physics yields a logical structure with for each possible physical 'environment' (e.g., a measurement apparatus, a free or imposed evolution, interaction in the presence of another system, etc.) the following connectives:

i. two implications $(- \stackrel{e}{\rightarrow} -)$ and $(- \stackrel{e}{\leftarrow} -)$ that extend the physical content of propagation of (physical) properties and backward causal assignment, and,
ii. two corresponding adjoint tensors $(- \otimes_e -)$ and $(-{}_e\otimes-)$ of which one is commutative and one is not.

This, since the Sasaki adjunction encodes causal duality, then establishes our claim made in the title concerning the induced dynamic implications by the Sasaki hook. In the 'static limit', i.e., 'freezed dynamics' with respect to some preferred referential frame for space-like properties, this structure yields an intuitionistic logic equipped with the above mentioned operational resolution as an additional operation, and both the hooks $(- \stackrel{e}{\rightarrow} -)$ and $(- \stackrel{e}{\leftarrow} -)$ collapse into the [static] Heyting implication, and the tensors $(- \otimes_e -)$ and $(-{}_e\otimes-)$ become binary conjunction.[9] We also recall here the following spin-off from all the above (for details we refer to corresponding cited papers):

i. A proof of linearity for deterministic evolution and for the Hilbert space tensor product as a description of quantum compoundness (Faure *et al.*, 1995; Coecke, 2000).
ii. A generalized notion of linearity for indeterministic transitions that saturates into ordinary linearity in the deterministic case (Coecke and Stubbe, 1999; Coecke *et al.*, 2001).
iii. A counter example to van Benthem's (1991, 1994) 'general dynamic logic in terms of relational structures': relational inverses have not necessarily any physical significance for non-classical systems (Coecke *et al.*, 2001b).

We will proceed as follows in this paper: Since we feel very strong about the fact that quantum logicality cannot be treated as a purely mathematical matter without specifying what one is actually talking about and that in every other case it might even be better to abandon the word quantum (at least as a reference to physics) in ones discourse, we provide in the next section an outline of the

[9] The multiplicative fragments respectively provide a commutative quantale and dual non-commutative quantale semantics (Coecke, 2002b; Coecke *et al.*, 2001b; Smets, 2001).

primitive physical notions from which we derive our formal notions.[10] Next, we recall some mathematical preliminaries required for this paper including Galois adjoints, Heyting algebras and the Sasaki adjunction itself. In Section 4, besides briefly recalling the results in Coecke (2002a), we discuss the Sasaki hook in perspective of these, in particular we argue that any true implicative connective on the lattice of properties of a quantum system has to be external and as such cannot be the Sasaki hook. In the fifth section, besides explaining causal duality and as such, the true significance of the Sasaki adjunction, we introduce 'dynamic causal relations' which express the intuitive contents of the Sasaki adjunction in an alternative way. These relations form the core of our approach in the sixth section where the formal content of the Sasaki adjunction will be implemented in the framework of Dynamic (Operational) Quantum Logic (DOQL)—see also Coecke (2002b), Coecke *et al.* (2001b) and Smets (2001). Our analysis in this paper ends with an overview of the dynamic implications $(- \stackrel{\varphi_a}{\rightarrow} -)$ and $(- \stackrel{\varphi_a}{\leftarrow} -)$ which we can deduce from the Sasaki adjunction. Finally, Section 7 points to the possible impact of our approach on the field of quantum logic and opens new perspectives to be elaborated in the future.

2. WHAT QUANTUM LOGICALITY CAN BE ABOUT

We claim that it makes no sense to discuss quantum logicality without specifying what the elements in the considered lattice physically stand for. Indeed, nonsense arguments, as for example indicated in Foulis and Randall (1984) and Piziak (1986), emerge due to conceptual mixup.[11] See also Smets (2001 § 6) for a more general survey on misunderstandings and misconceptions on physical logicality. To situate our perspective clearly we will recall here two major (well-defined) perspectives which are, though essentially different respectively being ontological (Jauch and Piron, 1969; Piron, 1976) and empirical (Foulis and Randall, 1972; Randall and Foulis, 1973), not at all exclusive (Foulis *et al.*, 1983), but which give rise to different mathematical structures —see for example Coecke *et al.* (2000) for an overview and Moore (1999) and Wilce (2000) for recent surveys respectively on the Jauch–Piron and the Foulis–Randall perspective. How can one theoretically approach the behavior of a physical system? As philosophers know very well (to whom physicist,

[10] Obviously, there is something to say for the use of the word quantum referring to a domain of mathematics that studies structures inspired on particular formal features of the quantum mechanical formalism such as non-distributivity, but this still remains pure mathematics in absence of an outline of the primitive physical notions from which one derives formal notions such as order, bounds and in particular of the significance of elements in any considered set on which one defines these relations and connectives. In this context, for a recent survey of general operational quantum logic we refer to Coecke *et al.* (2000).

[11] We rather not refer to the papers containing mathematical/conceptual flaws but give credit to those who tackled them.

however, in general don't pay much attention)[12] there are different answers to this question. As such, any approach requires a subtle specification of what the primitive notions are one starts from. In Foulis and Randall (1972) and Randall and Foulis (1973) one considers a notion to which we prefer to refer to as "observed events that reflect something about the system's qualities," where in Jauch and Piron (1969) and Piron (1976) one considers "qualities of the system that cause certain events to occur," depending on the particular environment (e.g., presence of a measurement device). As we know from quantum mechanics, the state of the system in general doesn't determine the outcome of a measurement, and, an event provoked by a measurement actually changes the system's qualities. As such, it comes as no surprise that these perspectives yield different mathematical structures. To a certain extend one could say that both in the Jauch–Piron and Foulis–Randall perspective, we are interested in how the system interacts with its environment, though in the first case from the 'system's perspective' where in the second case we rather consider the 'environments perspective,' including the physicist that effectuates the experiments, or in other words, an endo- versus an exo-perspective—see also Coecke (2002b) for a discussion on this matter, slightly deviating from the original Jauch–Piron approach allowing some additional flexibility in view of actual applications. Obviously, since the Foulis–Randall perspective is an exo-perspective, the measurements are made explicit within the formalism. Their formalism is indeed essentially about how the system's behavior is reflected through measurements, without specifying the behavior itself. In the Jauch–Piron perspective, where we focus on the system's behavior itself this is a somewhat more subtle matter. Since it adopts an endo-perspective, the measurement is not a priori part of the 'universe of discourse.' Therefore it will be incorporated in a conditional way, explicitly as "a system in a particular realization p, i.e., state, possesses a quality a if it is the case that: whenever it (in realization p) is within environment e_a then it causes phenomenon α_a to happen" and it is by this statement that we identify a particular quality of the system[13]—this explicit consideration of the environment (or context),

[12] See for example Rovelli (1999) who backs us up on this: "I am convinced of the reciprocal usefulness of a dialog between physics and philosophy (Rovelli, 1997). This dialog has played a major role during the other periods in which science faced foundational problems. In my opinion, most physicists underestimate the effect of their own epistemological prejudices on their research [. . .] On the one hand, a more acute philosophical awareness would greatly help the physicists engaged in fundamental research: Newton, Heisenberg and Einstein could not have done what they have done if they were not nurtured by (good or bad) philosophy."

[13] Note here also that "whenever the system is within environment e_a then it causes phenomenon α_a to happen" corresponds with Piron's "whenever a definite experimental project is effectuated, we obtain a positive outcome with certainty" (Piron, 1976; Moore, 1999) where the definite experimental project includes both a physical procedure, say placing the system within the environment e_a, and specification of what is a positive answer to this procedure, say phenomenon α_a happens. By referring to a causal connection, we aim to avoid the confusion raised by use of the notion 'certainty' in Piron's formulation. One could also more naively say that Piron's formulation is an active one (from

even in the system's endo-perspective, is what gives the operational flavor to this approach.[14]

Before we continue, let us first recall some basic order theoretical notions. A *complete lattice* is a bounded partially ordered set $(L, \leq, 0, 1)$ which is such that every subset $A \subseteq L$ has a greatest lower bound or meet $\bigwedge A$. It then follows that every subset $A \subseteq L$ also has a smallest upper bound or join $\bigvee A$ via *Birkhoff's theorem*:

$$\bigvee A = \bigwedge \{b \in L | \forall a \in A : b \geq a\}. \tag{1}$$

If the bounded poset $(L, \leq, 0, 1)$ only admits finite greatest lower bounds and finite least upper bounds, we call it a *lattice*. In case it has only finite greatest lower bounds and not necessarily least upper bounds we call it a *meet-semilattice*. A first main example of a complete lattice is the lattice $L_{\mathcal{H}}$ of closed subspaces of a Hilbert space $\mathcal{H}$, ordered by inclusion, or, isomorphically, the lattice of orthogonal projectors on this Hilbert space, ordered via $P_A \leq P_B \Leftrightarrow P_B \circ P_A = P_A \circ P_B = P_A$, i.e., if and only if we have $A \subseteq B$ for the corresponding subspaces (Dunford and Schwartz, 1957 § VI.3). In the closed subspace perspective meets correspond to intersection and joins to closed linear span. In the projector perspective it is harder to grasp the operations meets and join, since they only can be expressed in a simple tangible way in case of commuting projectors (Dunford and Schwartz, 1957 § VI.3). A second example is the powerset $\mathrm{P}(X)$ of any set X, i.e., the set of subsets of this set, ordered again by inclusion and meets and joins are respectively intersection and union. Orthomodular lattices generalize these two cases of the Hilbert space projection lattice and the powerset of a set. Recall here that an *orthomodular lattice* is a lattice that goes equipped with an *orthocomplementation*$' : L \to L$, defined by $a \leq b \Rightarrow b' \leq a'$, $a \wedge a' = 0$, $a \vee a' = 1$ and $a'' = a$, and which is such that $a \leq b$ implies $a \vee (a' \wedge b) = b$. Alternative characterizations of orthomodularity can be found in Section 3 of this paper. One verifies that every modular ortholattice is also an orthomodular lattice, and for that reason one refers to the additional property an orthomodular lattice has compared with an ortholattice as weak modularity.[15] For ortholattices, we have as such the following

a physicist's perspective) where our's is a passive one. Again, by the passive formulation we avoid any connotation with some role that is in many interpretations of quantum theory ascribed to the so-called 'observer.'

[14] Note that operationalism has here nothing to do with instrumentalism. In Piron's formulation the tendency towards an instrumentalist interpretation is, however, a bit stronger due to the explicit presence of 'definite experimental projects'. By considering general environments instead of specific physical procedures, we hope to avoid some confusion and eliminate the link to P. W. Bridgman's operationalism since in our case, physical qualities have an extension in reality and are not by means of definitions reducible to sets of procedures—see Smets (2001§ 1).

[15] You have reason to be confused here. However, an orthomodular lattice is in general not modular. Clearly a case of bad terminology, due to some formal confusion at the early development of the subject, something what most probably did not contribute to its general appreciation.

hierarchy Distributive $\Longrightarrow$ Modular $\Longrightarrow$ Weakly Modular, or, in terms of objects, **BoolAlg** $\subset$ **MOL** $\subset$ **OML**. We refer to Bruns and Harding (2000) for a recent survey on algebraic aspects of this matter. Finally recall that both examples considered above are examples of so-called *atomistic lattices* respectively having the one-dimensional subspaces $\Sigma_{\mathcal{H}}$ and the singletons $\{\{x\}|x \in X\}$ as atoms, explicitly,

$$\forall A \in L_{\mathcal{H}} : A = \bigvee_{\mathcal{H}} \{\mathrm{ray}(\psi) \in \Sigma_{\mathcal{H}} |\, \mathrm{ray}(\psi) \subseteq \mathrm{A}\}$$

$$\text{and} \quad \forall T \in \mathrm{P}(X) : T = \bigcup \{\{x\}|x \in T\}, \tag{2}$$

thus satisfying the general *atomisticity* condition $\forall a \in L : a = \bigvee\{p \in \Sigma | p \leq a\}$, where Σ denotes the atoms of L, i.e., $p \in \Sigma$ if and only if $\forall a \in L : a \leq p \Rightarrow a \in \{0, p\}$.

Now, coming back to the two perspectives on logicality mentioned above, we will "initially" take the endo-perspective, and look at the true 'proper' qualities of the system, to which we will refer briefly as *properties*. Later in the paper, the exo-perspective will enter naturally when defining logical hooks. The resulting structure will as such incorporate both! So in this paper a property is definitely not to be envisioned merely as an observed quality/quantity, since that would be an event of the Foulis–Randall perspective. We as such do assume a form of realism in the sense that properties do exist in the absence of a measurement.[16] For example, in the dark, one could attribute the property referred to as 'red' to an object which is such that, "whenever there is a white light source brought in its environment that shines on it, it radiates red light." Note here that we implicitly assume a system to be well-specified. Depending on its possible realization p (say *state*), the system possesses different properties L_p, referred to as the *actual properties* for that particular realization p. The collection of all properties that the system can possess within the boundaries of its domain of specification, all the corresponding realizations themselves being denoted as Σ (any other realization will be considered as destruction of the system), will be denoted by L. This set L goes naturally equipped with a partial order in terms of "actuality of $a \in L$ implies actuality of $b \in L$," i.e., for any (fixed) state we have that: if it is the case that "whenever it (...) is in environment e_a then it causes phenomenon α_a to happen," then this implies that "whenever it (...) is in environment e_b then it causes phenomenon α_b to happen." Denoting "a is actual in state p" as $p \prec a$ this formally becomes

$$(a \leq b) \Longleftrightarrow (\forall p)(p \prec a \Rightarrow p \prec b). \tag{3}$$

[16] Reality is obviously in no way to be understood as synonym for 'locality and non-contextuality' as it is sometimes the case in some (from a philosophical perspective) slightly naive discourses on philosophy of physics.

Moreover, this poset is closed under 'conjunctions' $\bigwedge A$ of properties $A \subseteq L$ where actuality of $\bigwedge A$ stands for "any $a \in A$ is actual," what actually means that, for each $a \in A$, whenever the system (...) is within environment e_a then it causes phenomenon α_a to happen.[17] So we consider here a not fully specified environment in order to establish, slightly abusively, a disjunction of environments $\{e_a\}_{a \in A}$ (and corresponding phenomena $\{\alpha_a\}_{a \in A}$).[18] The feature that distinguishes quantum systems from classical systems is the fact that we cannot define a disjunction of a collection A in this way. Given $A \subseteq L$, the statement "some $a \in A$ is actual" would require a simultaneity, or again slightly abusively, a conjunction of environments what conflicts with quantum theory where we have incompatibility of measurements corresponding to non-commuting self-adjoint operators. The conjunction $\bigwedge$ defined above provides L with a complete lattice structure, where, as already mentioned, the corresponding joins have not necessarily a disjunctive significance.[19] In particular can there be properties $b \in L$ that do not imply that some $a \in A$ is actual but that do imply the join $\bigvee A \in L$ to be actual, the so-called *superposition principle* of quantum theory—see Aerts (1981) and Coecke (2002a) for a rigorous discussion on this matter. Let us stress that at this point, as argued in Coecke *et al.* (2001a), the full physically derivable logical content that emerges from our operational setting consists of a consequence relation[20] $\vdash \subseteq \mathrm{P}(L) \times L$, that extends the lattice ordering $\leq \subseteq L \times L$ exploiting *conjunctivity*, i.e.,

$$\forall a \in A : \text{``}a \text{ is actual''} \iff \text{``}\bigwedge A \text{ is actual,''} \tag{4}$$

since this allows to transcribe the set $\{\text{``}a \text{ is actual''} \mid a \in A\}$ as "$\bigwedge A$ is actual" it

[17] Note here that contrary to a Tarskian perspective where one abstracts over the true sense of meets, we give a particular operational significance to it. Conjunctivity is in a sense "conjunctivity with respect to actuality," i.e., with respect to "causing phenomena α_a for $a \in A$ to happen whenever (...)." See for example Girard, Lafont and Taylor (1989) for a survey of some similar operational considerations on connectives in computation and proof theory, where one focuses in particular on the 'dynamics' underlying proofs and programs.

[18] Or, in Piron's terms "choose any $a \in A$ and place the system in e_a," i.e., a choice of environment. Again, in order to avoid any cognitive connotation, we prefer to avoid the word 'choice' (although, we don't see any a priori problem in its use).

[19] 'Disjunctive' to be seen again in terms of "disjunctive with respect to actuality."

[20] For the sake of the argument, we initially introduce here a Tarskian notion of consequence relation, i.e., following Tarski (1936, 1956). As such it can be seen as a binary relation on sets of formulas which satisfies reflexivity, monotonicity and transitivity. Later on we will extend this notion of consequence relation to allow multiple conclusions—see eq. (23)—following the ideas of D. Scott. Note, however, that in contemporary literature this type of consequence relation is often replaced by a 'weaker one' in the sense that substructural logicians prefer to work with multisets and/or non-monotonic logicians drop the monotonicity condition—for more details on this matter we refer to Avron (1994).

justifies setting [21]

$$a, \ldots (a \in A) \vdash b \Longleftrightarrow \bigwedge A \vdash b \Longleftrightarrow \bigwedge A \leq b. \tag{5}$$

We can introduce a (semantic) satisfaction relation $\models \subseteq \Sigma \times L$, exactly being the actuality relation $\prec$ between states and properties—we prefer to have this double use of notation $\prec$ and $\models$ to stress whether we are either talking about the physical content or the derived logicality. As such satisfaction and consequence are for single assumptions related by

$$a \vdash b \Longleftrightarrow (\forall p)(p \models a \Rightarrow p \models b) \tag{6}$$

and thus in general we have that $a, \ldots (a \in A) \vdash b \Longleftrightarrow (\forall p)((\forall a \in A : p \models a) \Rightarrow p \models b)$. For transparency of the argument below, we essentially consider single properties as arguments. The (hypothetical!) existence of some implication connective $(- \to -) : L \times L \to L$ would at least require that it satisfies the so-called 'minimal implicative condition', for single assumptions being $a \vdash b \Longleftrightarrow \vdash (a \to b)$, such that it extends the physically derivable implication relation encoded as the lattice ordering, what transcribes in lattice and state terms respectively as

$$\begin{aligned} a \leq b \ \Longleftrightarrow (a \to b) = 1 \quad \text{and} \quad \forall p \in \Sigma : (p \prec a \Rightarrow p \prec b) \\ \Longleftrightarrow \forall p \in \Sigma : p \prec (a \to b). \end{aligned} \tag{7}$$

However, validity of *deduction* moreover transcribes as (*sensu* Gentzen's sequent calculus) $\{a, c\} \vdash b \Rightarrow c \vdash (a \to b)$, or exploiting conjunctivity, $a \wedge c \vdash b \Rightarrow c \vdash (a \to b)$, what transcribes in lattice terms as

$$a \wedge c \leq b \Longrightarrow c \leq (a \to b). \tag{8}$$

Note here that the minimal implicative condition is actually a weakened form of the deduction theorem. Validity of both *modus ponens* $c \vdash (a \to b) \Rightarrow \{c, a\} \vdash b$ assures the converse implication, i.e.,

$$a \wedge c \leq b \Longleftarrow c \leq (a \to b). \tag{9}$$

We will come back to this point in the next section after recalling adjointness.

3. SASAKI ADJUNCTION

We recall some basic features of Galois adjoints. A more detailed survey of Galois adjoints can be found in Erné *et al.* (1993) and for Galois adjoints in a

[21] One could say that the notation $a, \ldots (a \in A)$ for representing actuality of each member in A, i.e., $\forall a \in A$: "a is actual" could be simplified by writing down A, since in general in sequent calculus a list of assumptions on the left of $\vdash$ always has to be interpreted conjunctively, i.e., as identifiable with the meet. However, further we will consider collections $A \subseteq L$ in terms of $\exists a \in A$: "a is actual" and they will also appear on the left of $\vdash$ since we will consider them as primitive propositions. The notion $a, \ldots (a \in A)$ is as such required to avoid confusion.

more physical perspective we refer to Coecke and Moore (2000). A pair of maps $f^* : L \to M$ and $f_* : M \to L$ between posets L and M is *Galois adjoint*, denoted by $f^* \dashv f_*$, if and only if

$$f^*(a) \leq b \Leftrightarrow a \leq f_*(b). \tag{10}$$

Stressing the mathematical importance of adjoints, we respectively quote the co-father of category theory S. Mac Lane and logician R. Goldblatt (Goldblatt, 1984 p. 438):

> "... adjoints occur almost everywhere in many branches of mathematics. ... a systematic use of all these adjunctions illuminates and clarifies these subjects."
>
> "The isolation and explication of the notion of adjointness is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas."

One could even say that where in the beginning days of category theory, the claim was made that it are the functors and natural transformations that constitute the core of category theory rather than the categories themselves, that it are actually the adjunctions that provide the true power. Coming back to eq.(10), in the case that f^* and f_* are inverse, and thus L and M isomorphic, the above inequalities saturate in equalities. As we show below, the notion of Galois adjoint retains some essential uniqueness properties of inverses. Whenever $f^* \dashv f_*$ then f^* preserves all existing joins and f_* all existing meets. This means that for a Galois adjoint pair between complete lattices, one of these maps preserves all meets and the other preserves all joins. Conversely, for L and M complete lattices, any meet preserving map $f_* : M \to L$ has a unique join preserving left Galois adjoint and any join preserving map $f^* : L \to M$ a unique meet preserving right Galois adjoint, respectively,

$$f^* : a \mapsto \bigwedge\{b \in M | a \leq f_*(b)\} \quad f_* : b \mapsto \bigvee\{a \in L | f^*(a) \leq b\}. \tag{11}$$

Thus, it follows that there is a one-to-one correspondence between the join preserving maps, between complete lattices and the meet preserving maps in the opposite direction, this so-called 'duality' being established by Galois adjunction. One also verifies that eq.(10) is equivalent to

$$\forall a \in L : a \leq f_*(f^*(a)) \quad \text{and} \quad \forall b \in M : f^*(f_*(b)) \leq b. \tag{12}$$

Considering pointwise ordering of maps, i.e., for $f, g : L \to M$, $f \leq g \Leftrightarrow \forall a \in L : f(a) \leq g(a)$ we can write the above as $id_L \leq f_* \circ f^*$ and $f^* \circ f_* \leq id_M$ where id_L and id_M are the respective identities on L and M. Now, coming back to eq.(8) and (9) of the previous section one sees that they define, when both of them are valid, an adjunction $(a \wedge -) \dashv (a \to -)$ for all $a \in L$. This particular property, i.e., the existence of a hook that acts as a right adjoint to the parameterized action of the meet, actually defines a Heyting algebra, a type of lattice to which we turn our attention now.

Let us recall some features of Heyting algebras—for a recent survey see for example Borceux (1994). A *Heyting algebra* is a lattice $(H, \wedge, \vee)$ equipped with an additional binary operation $(- \to -) : H \times H \to H$ that satisfies $a \wedge b \leq c \Leftrightarrow a \leq (b \to c)$, i.e., after exchanging a and b and applying commutativity of $(- \wedge -)$, the action of the meet is indeed left adjoint to the action of the hook, explicitly we have $(a \wedge -) \dashv (a \to -)$ for all $a \in H$. As such, a complete Heyting algebra encodes those lattices in which we encode validity of modus ponens and deduction in a semantical way. Also following from this adjointness, in any Heyting algebra $(a \wedge -)$ preserves existing joins, explicitly, $a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$, so it turns out that a Heyting algebra is always distributive . In fact, any Boolean algebra, e.g., $\mathrm{P}(X)$ for any set, turns out to be a Heyting algebra with $(a \to b) = {}^c a \vee b$, where ^{c} denotes complementation, $(a \to b)$ then being logically interpretable as "(not a) or b." A so-called *pseudo-complement* can be defined on any Heyting algebra as $\neg(-) : H \to H : a \mapsto (a \to 0)$ given a lower bound 0 of H. It then, however, turns out that contrary to a Boolean algebra we in general do not have that $\neg a \vee a = 1$ given an upper bound 1 of H, what justifies the notion of pseudo-complement. In this paper we will only consider complete Heyting algebras, where a Heyting algebra is complete if and only if the underlying lattice is complete. Now, since $(a \wedge -)$ preserves the joins of all subsets of a complete Heyting algebra H, we obtain a stronger form of distributivity namely $a \wedge (\bigvee B) = \bigvee_{b \in B}(a \wedge b)$. In fact, this *complete distributivity* now fully determines the complete Heyting algebra structure in the sense that for all $a \in H$ the map $(a \wedge -) : H \to H$ preserves all joins so it has a unique right adjoint $(a \to -) : H \to H$. So complete Heyting algebras are complete lattices where the join of all subsets is completely distributive over binary meets. We will now present an example of a complete Heyting algebra which is in general not a Boolean algebra. Let L be any poset and set $\downarrow a := \{b \in L | b \leq a\}$ for $a \in L$ and introduce a *downset* or *order ideal* as any set of the form $\downarrow[A] := \{b \in L | \exists a \in A : b \leq a\} = \bigcup_{a \in A} \downarrow a$ with $\emptyset \neq A \subseteq L$, i.e., I is an order ideal if and only if $I \neq \emptyset$ and $a \leq b \in I$ implies $a \in I$. Order ideals of the form $\downarrow a$ for $a \in L$ are called *principal ideals*. It then turns out that the collection of non-empty downsets $\mathrm{I}(L) := \{\downarrow[A] | \emptyset \neq A \subseteq L\}$ constitutes a complete Heyting algebra. Indeed, since unions and intersections of downsets are again downsets, $\mathrm{I}(L)$ is closed under unions and intersections from which it follows that they respectively constitute the join and meet in $\mathrm{I}(L)$. Distributivity of $\mathrm{I}(L)$ is as such inherited from that of $\mathrm{P}(L)$. Using eq.(11) we can now compute the corresponding Heyting algebra hook

$$(B \to_{\mathrm{I}(L)} C) = \bigcup \{A \in \mathrm{I}(L) | A \cap B \subseteq C\} = \{a \in L | \forall b \in B : a \wedge b \in C\}. \tag{13}$$

As pseudo-complement we obtain $\neg B = (B \to_{\mathrm{I}(L)} 0_{\mathrm{I}(L)}) = \{a \in L | \forall b \in B : a \wedge b = 0_L\}$. So in general we indeed do not have $B \bigcup \neg(B) = 1_{\mathrm{I}(L)}$. The most simple example of a Heyting algebra which is not Boolean is a three element chain

$\{0 < a < 1\}$. In particular are the downsets of any chain isomorphic to the chain itself, establishing the claim that downsets in general don't constitute a Boolean algebra. Another example are the open sets of a topological space ordered by inclusion.

Recall that for the lattice of closed subspaces of a Hilbert space the *Sasaki projection*

$$\varphi_A^* : L_{\mathcal{H}} \to L_{\mathcal{H}} : B \mapsto A \cap (A^\perp \vee_{\mathcal{H}} B) \tag{14}$$

exactly encodes the action of the orthogonal projector P_A that projects on the subspace A, so in particular we have for the action on rays that

$$\varphi_A^*(\mathrm{ray}(\psi)) = \mathrm{A} \cap (\mathrm{A}^\perp \vee_{\mathcal{H}} \mathrm{ray}(\psi)) = \mathrm{ray}(\mathrm{P_A}(\psi)) \tag{15}$$

where we identify $\mathrm{ray}(P_A(\psi))$ in case that $\psi \perp A$ with the zero-dimensional subspace. Moreover, for an arbitrary orthomodular lattice L, setting

$$\varphi_a^* : L \to L : b \mapsto a \wedge (a' \vee b) \quad \text{and} \quad \varphi_{a,*} : L \to L : b \mapsto a' \vee (a \wedge b), \tag{16}$$

for all $a \in L$, we have $\varphi_a^* \dashv \varphi_{a,*}$. Indeed, if $a \wedge (a' \vee b) \leq c$ then $a' \vee (a \wedge (a \wedge (a' \vee b))) \leq a' \vee (a \wedge c)$ where $b \leq a' \vee b = a' \vee (a \wedge (a' \vee b))$ since $a' \leq a' \vee b$, and analogously one proves the converse. This adjunction actually embodies why the *Sasaki hook* $(- \stackrel{S}{\to} \cdot) := \varphi_{(-),*}(\cdot)$ has been interpreted as an implication, since φ_a^* coincides with $(a \wedge -) : L \to L$, the classical projections, in the case that L is distributive since then $a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b) = 1 \vee (a \wedge b) = a \wedge b$.[22] In particular do we as such retain an adjunction of projection action and hook, mimicking the one that one has for complete Heyting algebras that embodies the validity of modus ponens and the kind of deduction theorem obtained by combining eq.(8) and (9). However, this in no way implies that all tools available in classical/intuitionistic logic will still be valid within this setting. Let us briefly outline how the minimal implicative condition and the adjointness for the Sasaki hook relate, both in the cases that we abstract over the explicit formulation of the Sasaki hook and the Sasaki projection, i.e., the case of a general abstract adjoint implication (Hardegree, 1979, 1981) on a bounded poset, and the case of them being explicitly defined on an ortholattice. We follow Coecke *et al.* (2001c). Let $\mathrm{J}(L)$ be the collection of isotone maps on a bounded poset L that admit a right adjoint. An adjoint implication is then defined by a map $\tilde{\varphi}^* : L \to \mathrm{J}(L) : a \mapsto \tilde{\varphi}_a^*$ that satisfies $\tilde{\varphi}_a^*(1) = a$. The parameterized right adjoint $(- \stackrel{\tilde{\varphi}}{\to} -) : L \times L \to L$, i.e., $\tilde{\varphi}_a^* \dashv (a \stackrel{\tilde{\varphi}}{\to} -) := \tilde{\varphi}_{a,*}$, is then to what we refer as the *adjoint implication*. The

[22] This view is obviously motivated by the fact that where for a Heyting algebra the actions $\{(a \wedge -) | a \in L\}$ can be envisioned as projections on a, for orthomodular lattices the Sasaki projections $\{\varphi_a^* | a \in L\}$ are the closed orthogonal projections in the Baer *-semigroup of L-hemimorphisms (Foulis 1960) which, as mentioned above, coincide in the case of the subspace lattice of a Hilbert space with the action of the closed projectors of the underlying Hilbert space. We also refer to Coecke and Smets (2000) for complementary details on this matter.

condition $\tilde{\varphi}_a^*(1) = a$ implies the minimal implicative condition via explicitation of the adjunction, i.e., $\tilde{\varphi}_a^*(1) \leq c \Leftrightarrow b \leq (a \stackrel{\tilde{\varphi}}{\to} c)$, for $b = 1$. One could as such state that because the Sasaki projections satisfy $\varphi_a^*(1) = a$, the Sasaki hook satisfies the minimal implicative condition. For an ortholattice it turns out that adjointness of the Sasaki hook and the Sasaki projection is equivalent to one side of the implicative condition, namely $(a \stackrel{S}{\to} x) = 1 \Rightarrow a \leq x$—notice that the other side is trivially satisfied for ortholattices since $a \leq x$ implies $a' \vee (a \wedge x) = a' \vee a = 1$. From this perspective one can say that the minimal implicative condition incarnates the fact that the Sasaki hook arises as the right adjoint of Sasaki projections. Moreover, these two alternative definitions are actually equivalent to the ortholattice being orthomodular, and as such provide alternative characterizations of orthomodularity, respectively one that can be written equationally, a rather logical one, and one in terms of an adjunction.

Proposition 3.1. *The following are equivalent for an ortholattice L:*

i. *L is orthomodular, i.e., $a \leq b$ implies $a \vee (a' \wedge b) = b$;*
ii. *For all $a \in L$ we have $(a \stackrel{S}{\to} x) = 1 \Rightarrow$ (or $\Leftrightarrow$)$a \leq x$;*
iii. *For all $a \in L$ we have $\varphi_a^*(-) \dashv (a \stackrel{S}{\to} -)$.*

By (i) $\Leftrightarrow$ (ii), one has a statement concerning logicality attributed to the Sasaki hook in terms of the minimal implicative condition, or equivalently, the Sasaki adjunction incarnates the utterance ‘orthomodular logic’.[23] As is reflected in the title and introduction of this paper, we do not follow this line of thought! For us, it is (i) $\Leftrightarrow$ (iii) that will provide a new interpretation of orthomodularity in terms of causal duality.

4. TRUE IMPLICATIVE QUANTUM LOGICALITY

In this section we essentially follow Coecke (2002a). As discussed in Section 2, the lattice of properties in general does not encode arbitrary disjunction, since otherwise, they would constitute the joins and as such all joins would be disjunctions, what is in general not the case. Let us first analyze what happens in a (dichotomic) perfect quantum measurement.[24] Consider a (dichotomic) perfect quantum measurement of the property $a \in L$ and correspondingly, its orthocomplement a'. Assuming that a property $b \in L$ is actual before the measurement, it follows, since the Sasaki projections encode projectors on subspaces, that after the measurement either $\varphi_a^*(b)$ or $\varphi_{a'}^*(b)$ is actual. Indeed, referring back to eq. (14)

[23] See also Moore (1993) on this matter.
[24] For the introduction of the respective concepts of (dichotomic) ideal measurement and (dichotomic) measurement of the first kind, and, conjointly, a (dichotomic) perfect measurement, we refer to Pauli (1958) and Piron (1976).

and (15), and recalling that dichotomic measurements are in quantum theory represented by self-adjoint operators with a binary spectrum, the projectors on the corresponding (mutually orthogonal) eigenspaces are then exactly encoded by φ_A^* and $\varphi_{A^\perp}^*$, where A and $A^\perp$ are the corresponding eigenspaces. Writing "we obtain either $\varphi_A^*(\text{ray}(\psi))$ or $\varphi_{A^\perp}^*(\text{ray}(\psi))$ as outcome state" then corresponds to an abstraction over the corresponding probabilistic weights of the two outcomes in a dichotomic measurement, focusing on the fact that whenever we are not in an eigenstate, there is an uncertainty on the outcome. Consequently, there is also an uncertainty on the corresponding 'change of state' (according to the projection postulate), and as such, an uncertainty on the corresponding 'change of actual properties.' We refer to this logical feature of quantum measurements as the 'emergence of disjunction in quantum measurements.' Writing this in a more formal, though intuitive way using a consequence 'symbol' we obtain:

$$\text{“}b \text{ actual”} \vdash_{\text{perf. meas. of}\{a,a'\}} \text{“}\varphi_a^*(b) \text{ actual”}\ \underline{\text{or}}\ \text{“}\varphi_{a'}^*(b) \text{ actual”} \tag{17}$$

or, when assuming the existence of an appropriate implicative connective $\longrightarrow_{\text{perf. meas. of}\{a,a'\}}$ that satisfies the corresponding minimal implicative condition this becomes via the corresponding weakened form of the deduction theorem:

$$\text{“}b \text{ actual”} \longrightarrow_{\text{perf. meas. of}\,\{a,a'\}} \text{“}\varphi_a^*(b) \text{ actual”}\ \underline{\text{or}}\ \text{“}\varphi_{a'}^*(b) \text{ actual.”} \tag{18}$$

Unfortunately, "$\varphi_a^*(b)$ actual" or "$\varphi_{a'}^*(b)$ actual," i.e., "a member of the pair $\{\varphi_a^*(b), \varphi_{a'}^*(b)\}$ is actual," is not encoded in the lattice of properties of a quantum system as an element since for example, taking $b = 1$, we have $\varphi_a^*(1) \vee \varphi_{a'}^*(1) = 1$ independent on a though the possible states the system can have—given that "$\varphi_a^*(b)$ actual" or "$\varphi_{a'}^*(b)$ actual" — definitely depend on a. Thus, it would make sense to have logical propositions that express disjunctions of properties since they emerge in quantum processes, in the endo-perspective. The question then arises whether we can extend L with propositions of the type "$\varphi_a^*(b)$ actual" or "$\varphi_{a'}^*(b)$ actual", or equivalently, "a member of $\{\varphi_a^*(b), \varphi_{a'}^*(b)\}$ is actual," without loosing the logicality encoded in the initial lattice of properties, i.e., the lattice order, and whether this can be done in a non-redundant, canonical or even mathematically universal way.[25]

A first candidate for encoding disjunctions would be the powerset $P(L)$. However, if $a \leq b$ we do not have $\{a\} \subseteq \{b\}$ so we do not preserve the initial logicality, or, otherwise stated, if $a < b$ then the *propositions* $\{a\}$ and $\{a, b\}$ ('read' $\{a, b\}$ as: either a or b is actual) mean the same thing, since actuality of b is implied by that of a . We can clearly overcome this problem by restricting to order ideals $I(L) := \{\downarrow[A] | A \subseteq L\} \subset P(L)$. However, we encounter a second problem. In case

[25] Note that the fact that the <u>or</u> that we obtain in a perfect measurement is exclusive does not have to be encoded explicitly since it is already captured by the orthocomplementation since we have $\varphi_a^*(b) \wedge \varphi_{a'}^*(b) = 0$ by $a \wedge a' = 0$ such that "both φ_b^* and $\varphi_{b'}^*$ are actual" is excluded—0 indeed encodes the 'absurd.'

the property lattice would be a complete Heyting algebra in which all joins encode disjunctions, then A and $\{\bigvee A\}$ again mean the same thing. As argued in Coecke (2002a), this redundancy is then exactly eliminated by considering distributive ideals $\mathrm{DI}(L)$ (Bruns and Lakser, 1970), that is, order ideals, that are closed under *joins of distributive sets* (abbreviated as *distributive joins*), i.e., if $A \subseteq I \in \mathrm{DI}(L)$ then $\bigvee A \in I$ whenever we have $\forall b \in L : b \wedge \bigvee A = \bigvee\{b \wedge a \mid a \in A\}$. For L atomistic and $\Sigma \subseteq L$, $\mathrm{DI}(L) \cong \mathrm{P}(\Sigma)$ which implies that $\mathrm{DI}(L)$ is a complete atomistic Boolean algebra (Coecke, 2002a). We can moreover provide, from a mathematical perspective, more rigorous reasoning which exhibits the canonical nature of this construction. Consider the following definitions for $A \subseteq L$:

i. $\bigvee A$ is called *disjunctive* iff ("$\bigvee A$ is actual" $\Leftrightarrow \exists a \in A$: "a is actual") ;[26]
ii. *Superposition states* for $\bigvee A$ are states for which "$\bigvee A$ is actual" while "no $a \in A$ is actual";
iii. *Superposition properties* for $\bigvee A$ are properties of which the actuality implies that "$\bigvee A$ is actual" (without being equivalent to $\bigvee A$), and, that can be actual while "no $a \in A$ is actual."

Extending the satisfaction relation encoding actuality by $p \models A \Longleftrightarrow \exists a \in A : p \models a$ allows us to set

$$(\forall p)\left(p \models \bigvee A \Leftrightarrow p \models A\right) \Longleftrightarrow \bigvee A \text{ is disjunctive} \tag{19}$$

$$\left(p \models \bigvee A \quad \text{and } p \not\models A\right) \Longleftrightarrow p \in \Sigma \text{ is a superposition state of } \bigvee A, \tag{20}$$

$$\left(b \dot{\vdash} \bigvee A \quad \text{and} \quad b \not\vdash A\right) \Longleftrightarrow b \in L \text{ is a superposition property of } \bigvee A. \tag{21}$$

where $a \dot{\vdash} b$ means $a \vdash b$ but $b \not\vdash a$.

Proposition 4.1. *If "existence of superposition states implies existence of superposition properties" then*

$$\bigvee A \textit{ disjunctive} \Longleftrightarrow \bigvee A \textit{ distributive}. \tag{22}$$

Proof: See Coecke (2002a). □

The necessity condition "existence of superposition states implies existence of superposition properties" can be illustrated by means of the following example. When considering the four element lattice $\{0 \leq a, a' \leq 1\}$ even then distributivity and disjunctivity are not necessarily equivalent, for example in the case that $\Sigma = \{p, q, r\}$ and $p \prec a$, $q \prec a'$, $r \prec 1$ and $r \not\prec a, a'$—the superposition state r for

[26] Compare this definition with the one of conjunctivity.

$a \vee a'$ has no corresponding superposition property. Note here that for the particular example of quantum theory the condition is trivially satisfied as it is the case for any atomistic property lattice.[27] Now, any complete lattice L has $\mathrm{DI}(L)$ of distributive ideals as its distributive hull (Bruns and Lakser, 1970), providing the construction with a (quasi-)universal property.[28] Moreover, $\mathrm{DI}(L)$ itself always proves to be a complete Heyting algebra and the inclusion preserves all meets and existing distributive joins. Thus, $\mathrm{DI}(L)$ encodes all possible disjunctions of properties, and moreover, it turns out that all $\mathrm{DI}(L)$-meets are conjunctive and all $\mathrm{DI}(L)$-joins are disjunctive—note again that this is definitely not the case in the powerset $\mathrm{P}(L)$ of a property lattice, since $\{a\} \cap \{b\} = \emptyset$ whenever $a \neq b$ independent of what $a \wedge b$ is. This means that we can 'extend' the consequence relation of eq. (5) to $\vdash \subseteq \mathrm{P}(\mathrm{DI}(L)) \times \mathrm{P}(\mathrm{DI}(L))$ respectively in terms of consequence and satisfaction as

$$A, \ldots (A \in \mathcal{A}) \vdash B, \ldots (B \in \mathcal{B}) \Longleftrightarrow \bigcap \mathcal{A} \vdash \bigvee_{\mathrm{DI}(L)} \mathcal{B} \Longleftrightarrow \bigcap \mathcal{A} \subseteq \bigvee_{\mathrm{DI}(L)} \mathcal{B}, \tag{23}$$

$$A, \ldots (A \in \mathcal{A}) \vdash B, \ldots (B \in \mathcal{B}) \Longleftrightarrow (\forall p)((\forall A \in \mathcal{A} : p \models A) \Rightarrow (\exists B \in \mathcal{B} : p \models B)) \tag{24}$$

where we recall that $a \leq b \Leftrightarrow \downarrow a \subseteq \downarrow b$ and as such $\downarrow \left(\bigwedge A\right) = \bigcap_{a \in A} \downarrow a$ encodes the properties within this set of propositions on properties—$A \subseteq L$ is here to be seen as just a set of properies without the disjunctive connotation of the distributive ideals in $\mathrm{DI}(L)$. As demonstrated in Coecke (2002a) it follows from all this that the object equivalence between:

i. complete lattices, and,
ii. complete Heyting algebras equipped with a *distributive closure*, i.e., it preserves distributive sets,

encodes a disjunctive representation for property lattices, where we, $\mathrm{DI}(L)$ being a complete Heyting algebra, do have a Heyting hook

$$(- \to_{\mathrm{DI}(L)} -) : \mathrm{DI}(L) \times \mathrm{DI}(L) \to \mathrm{DI}(L) : (B, C) \mapsto \bigvee_{\mathrm{DI}(L)} \{A \in \mathrm{DI}(L) | A \cap B \subseteq C\}, \tag{25}$$

[27] Whenever this condition is satisfied one can also construct a disjunctive extension, which is obviously not anymore the distributive extension, but something 'in between' the distributive extension and the downset completion where corresponding inclusions are injective order embeddings that preserve all meets and the bottom element (Coecke, 2002a).

[28] Injective hulls are actually not universal in a strictly categorical sense. However, it is possible to give a characterization of distributive hulls in terms of a so-called 'frame completion', which is a mono reflection and as such strictly universal. See for example Harding (1999) and Stubbe (2001).

where the explicit DI(L)-joins are given by

$$\bigvee_{\mathrm{DI}(L)} : \mathrm{P}(\mathrm{DI}(L)) \to \mathrm{DI}(L) : \mathcal{A} \mapsto \bigcap \left\{ B \in \mathrm{DI}(L) \,\middle|\, B \supseteq \bigcup \mathcal{A} \right\} \tag{26}$$

(meets are obviously intersections), and we will argue below that this hook is *implicative* sensu "with respect to actuality," i.e., in the sense that in the property lattice $\bigwedge$ was conjunctive and in the sense we defined disjunctivity when motivating the use of distributive ideals. One verifies that we actually obtain[29]

$$(B \to_{\mathrm{DI}(L)} C) = \{a \in L | \forall b \in B : a \wedge b \in C\}. \tag{27}$$

The true quantum features are (at this 'static' level) encoded in an *operational resolution*

$$\mathcal{R}_{\mathrm{DI}(L)} : \mathrm{DI}(L) \to \mathrm{DI}(L) : A \mapsto \downarrow \left(\bigvee_L A \right) \tag{28}$$

that recaptures statements expressing actuality of properties within the larger collection of propositions on actuality of them, and that only for classical systems becomes trivial, being the identity. The logical essence of this representation is such that, rather than seeing the shift "from classical to quantum" as a weakening of the property lattice structure from a distributive lattice to a non-distributive one, we envision this transition as going from a trivial additional operation on the propositions (which as a consequence in the classical case coincide with the properties) to a non-trivial one. Note that the non-distributive features are as such recaptured as the range of this additional operation $\mathcal{R}$, but they don't affect distributivity of the domain. Quantum logic becomes as such ordinary logic with an additional operation, a bit in the sense of modal logic. From a pragmatic formal attitude, this construction however seems to conflict with statements about the non-distributive nature of quantum theory, what, for some authors is exactly the essence of quantum logicality. In the quantum case, the non-distributivity does not come in within the ordering of propositions, but as the range of the operation $\mathcal{R}$ which acts on the propositions. Also the other axioms considered in axiomatic approaches, e.g., orthomodularity, have the same incarnation. We refer to Coecke (2002b) for a more elaborated discussion on the significance/conceptions of non-distributivity in the context of quantum theory.

Going back to the explicit construction of the Heyting hook for propositions on properties, it as such also turns out that the canonical implication on a lattice

[29] Note here that $\{a \in L | \forall b \in B : a \wedge b \in C\}$ indeed defines a member of DI(L). First, we have that $x' \leq x$ implies $x' \wedge b \leq x \wedge b$ for $x \in (B \to_{\mathrm{DI}(L)} C) = \{a \in L | \forall b \in B : a \wedge b \in C\}$ such that $x' \wedge b \in C$ since $C \in \mathrm{DI}(L)$. Next, for $X \subseteq (B \to_{\mathrm{DI}(L)} C)$ with X a distributive set, i.e., $\forall c \in L : c \wedge \bigvee X = \bigvee \{c \wedge x \mid x \in X\}$, one easily verifies that distributivity of X implies distributivity of $\{b \wedge x \mid x \in X\}$ and thus $(\bigvee X) \wedge b = \bigvee \{b \wedge x \mid x \in X\} \in C$ so $\bigvee X \in (B \to_{\mathrm{DI}(L)} C)$.

of properties is an external one that takes values in DI(L), namely

$$(- \to_L -) : \ L \times L \to \mathrm{DI}(L) : (b, c) \mapsto \{a \in L | a \wedge b \leq c\} \tag{29}$$

obtained by domain and codomain restriction of $(- \to_{\mathrm{DI}(L)} -)$. If and only if L is itself a complete Heyting algebra, then we can represent this external operation faithfully as an internal one by setting $(b \to c) := \bigvee(b \to_L c)$. In particular can our external implication arrow be defined by

$$a \wedge b \leq c \Longleftrightarrow a \in (b \to_L c), \tag{30}$$

as such in a more explicit manner expressing that it generalizes the implication that lives on a complete Heyting algebra where $a \in (b \to_L c)$ then coincides with $a \leq (b \to c)$. Within DI(L) we see that $(b \to_L c)$ is the set of properties whose actuality makes the deduction "if b is actual then c is actual" true, i.e., given $a \in (b \to_L c)$, then $\forall p \in \Sigma_a : (p \models b \Rightarrow p \models c)$ where $\Sigma_a := \{p \in \Sigma | p \models a\}$. In other words, $(- \to_L -)$ transcribes in terms of actuality the minimal requirement of any functional formal implication with respect to extensional quantification over the state set. Note that by constructing $(- \to_L -)$ via (slightly abusively) domain restriction from $\mathrm{DI}(L) \times \mathrm{DI}(L)$ to $L \times L$ we exhibit clearly that $(- \to_L -)$ can (again) be made internal via a domain extension, and, that this extension has physical significance and moreover preserves all the physically derivable logicality of L. As a statement: "$(- \to_L -)$ is the [closest you can get to] implication on the lattice of properties." In relation to the minimal implicative condition we obtain

$$L \in (a \to_L b) \Longleftrightarrow a \leq b \Longleftrightarrow \forall p \in \Sigma : (p \models a \Rightarrow p \models b) \tag{31}$$

for $(- \to_L -)$ whereas for $(- \to_{DI(L)} -)$ this 'extends' to

$$L = (A \to_{\mathrm{DI}(L)} B) \Longleftrightarrow A \subseteq B \Longleftrightarrow \forall p \in \Sigma : (p \models A \Rightarrow p \models B). \tag{32}$$

Note that $\to_{\mathrm{DI}(L)}$ as an operation, is the parameterized right adjoint with respect to the respective meet actions $\{(A \cap -) | A \in \mathrm{DI}(L)\}$. The above leads to a semantical interpretation of $(- \to_{\mathrm{DI}(L)} -)$ as $p \models (A \to_{\mathrm{DI}(L)} B) \Longleftrightarrow (p \models A \Rightarrow p \models B)$, or, equivalently,

$$\mu(A \to_{\mathrm{DI}(L)} B) = \{p \in \Sigma | p \models A \Rightarrow p \models B\}, \tag{33}$$

where, extending the usual *Cartan map* $\mu : L \to \mathrm{P}(\Sigma) : a \mapsto \{p \in \Sigma | p \prec a\}$, we define $\mu(A) := \bigcup \mu[A]$, the square brackets referring to pointwise application of μ, i.e., in semantical terms, $\mu(A) = \{p \in \Sigma | p \models A\}$. Note at this point that there is indeed a duality in representing propositions in terms of DI(L) or in terms of a particular subset F(Σ) of P(Σ) defined as $\mathrm{F}(\Sigma) := \{\mu(A) | A \in \mathrm{DI}(L)\}$, which, for atomistic L , turns out to be P(Σ) itself. In more syntactical terms, i.e. without referring to the state space interpretation, adjointness allows us explicitly to restate

$(- \rightarrow_{\mathrm{DI}(L)} -)$ as[30]

$$(A \rightarrow_{\mathrm{DI}(L)} B) = \bigvee_{\mathrm{DI}(L)} \{C \in \mathrm{DI}(L) \mid \forall D \vdash C : (D \vdash A \Rightarrow D \vdash B)\} \qquad (34)$$

$$(A \rightarrow_{\mathrm{DI}(L)} B) = \bigvee_{\mathrm{DI}(L)} \{C \in \mathrm{DI}(L) \mid \forall d \in C : (d \in A \Rightarrow d \in B)\} \qquad (35)$$

$$(A \rightarrow_{\mathrm{DI}(L)} B) = \{c \in L \mid \forall d \leq c : (d \in A \Rightarrow d \in B)\}, \qquad (36)$$

for the latter, rather than explicitly showing that is indeed a distributive ideal such that we can drop the corresponding closure, we can use eq. (27) and verify straightforwardly that

$$\forall a \in A : a \wedge c \in B \Longleftrightarrow \forall d \leq c : (d \in A \Rightarrow d \in B). \qquad (37)$$

As such, for properties we obtain, syntactically,

$$(a \rightarrow_L b) = \{c \in L \mid \forall d \vdash c : (d \vdash a \Rightarrow d \vdash b)\}. \qquad (38)$$

How does the Sasaki hook relate to this implication? Since $\varphi_a^*(b) = a \wedge (b \vee a') \geq a \wedge b$ and thus $\varphi_-^*(-) \geq (- \wedge -)$ pointwisely—recall here that $(- \stackrel{L}{\rightarrow} -)$ arises as domain restriction of the right adjoint of the action of the $\mathrm{DI}(L)$-meets, the latter encoding for properties the L-meet in terms of intersection of principal ideals. For the corresponding adjoints of the actions we have $a \wedge (a \stackrel{S}{\rightarrow} b) = a \wedge (a' \vee (b \wedge a)) = \varphi_a^*(b \wedge a) = b \wedge a \leq b$ and thus $(a \stackrel{S}{\rightarrow} b) \in \{c \in L | a \wedge c \leq b\} = (a \rightarrow_L b)$, or, differently put, $\downarrow(- \stackrel{S}{\rightarrow} -) \leq (- \rightarrow_L -)$ pointwisely. In terms of actions this gives us for the corresponding adjoint pairs $(a \wedge -) \dashv (a \rightarrow_L -)$ — with slight abuse of notation, see eq.(30)—and $\varphi_a^*(-) \dashv (a \stackrel{S}{\rightarrow} -)$

$$(a \rightarrow_L -) \geq \downarrow (a \stackrel{S}{\rightarrow} -) \quad \text{and} \quad (a \wedge -) \leq \varphi_a^*(-), \qquad (39)$$

expressing reversal of pointwise order by adjunction. Thus, from a semantical perspective, $\downarrow(- \stackrel{S}{\rightarrow} -) : L \times L \rightarrow \mathrm{DI}(L)$ is a restriction of the (static) implication $(- \rightarrow_L -)$. So, if the Sasaki adjunction doesn't encode a 'real' implication, what does it do? This will be explained in the next section.

5. THE SASAKI ADJUNCTION INCARNATES CAUSAL DUALITY

Causal duality has been derived in Coecke *et al.* (2001) inspired on derivations in Faure *et al.* (1995). Rather than giving a full derivation, we sketch a more intuitive way of looking at the obtained results. Assume (so we don't give a full proof here)

[30] Since $A \subseteq B \Longleftrightarrow \forall a \in A : a \in B \Longleftrightarrow \forall p \models A : p \models B \Longleftrightarrow (\forall p)(p \models A \Rightarrow p \models B)$, due to the equivalence induced by eq. (6), we have $C \subseteq (A \rightarrow_{\mathrm{DI}(L)} B) \Longleftrightarrow (C \cap A) \subseteq B \Longleftrightarrow (\forall p)(p \models C \cap A \Rightarrow p \models B) \Longleftrightarrow (\forall p)(p \models C, p \models A \Rightarrow p \models B) \Longleftrightarrow \forall p \models C : (p \models A \Rightarrow p \models B)$,where again eq. (6) allows expressing this in terms of $D \vdash C$ rather than $p \models C$.

for a system placed in an environment e, e.g., an environment e_a sensu Section 2,[31] during a time interval $[t_1, t_2]$ (which can be envisioned as being infinitesimal) that there exist the maps:[32]

i. 'Propagation of properties' $e^* : L_1 \to L_2$ that assigns to any property $a_1 \in L_1$ the strongest property $e^*(a_1) \in L_2$ of which actuality is implied at time t_2 due to actuality of a_1 at time t_1;
ii. 'Causal assignment of properties' $e_* : L_2 \to L_1$ that assigns to any property $a_2 \in L_2$ the weakest property $e_*(a_2) \in L_1$ whose actuality at time t_1 guarantees actuality of a_2 at time t_2.

Since, given $a_2 \in L_2$, $e_*(a_2) \in L_1$ guarantees actuality of a_2 at time t_2, $e_*(a_2)$ has to propagate to a property that is stronger (or equal) than a_2 and as such $e^*(e_*(a_2)) \leq a_2$. Analogously, given $a_1 \in L_1$, since it propagates into $e^*(a_1)$, actuality of a_1 at t_1 guarantees actuality of $e^*(a_1)$ at t_2 and as such $a_1 \leq e_*(e^*(a_1))$. Thus, from $e^*(e_*(a_2)) \leq a_2$ and $a_1 \leq e_*(e^*(a_1))$ we obtain $e^* \dashv e_*$, and this adjunction is what we refer to as causal duality. The generality of the principle lies in the fact that besides applying to temporal processes it also applies to compoundness (Coecke, 2000).[33]

We started the first paragraph of the previous section with a discussion on (dichotomic) perfect quantum measurements with the aim to exhibit the emergence of disjunction. In view of Section 2 we can denote the corresponding environment that provokes such a measurement, i.e., the presence of the corresponding measuring device, as $\varphi_{\{a,a'\}}$. In the following paragraphs, we then additionally argued that the disjunctive extension has the extra advantage that it allows us to encode an external implicative hook on $L \times L$ which then extends to an internal implication on the whole of $\mathrm{DI}(L) \times \mathrm{DI}(L)$. Thus, the use of the disjunctive extension for representing quantum systems goes beyond representing the emergent disjunction in the sense that it has also a pure logical motivation in terms of envisioning (static) quantum logicality as ordinary logicality with the additional presence of a non-trivial operational resolution $\mathcal{R} : \mathrm{DI}(L) \to \mathrm{DI}(L)$. For the particular case of a perfect quantum measurement, we are going to restrict us now to the specific example where we consider a transition only provided a certain positive outcome is obtained, say a for simplicity, what actually means that whenever the system

[31] And for simplicity assumed to be non-destructive.

[32] The existence can be proved—see Coecke *et al.*, (2001) and Faure *et al.*, (1995).

[33] We want to stress here that causal duality actuality allows us to prove things and is such is not just a fancy way of writing things down. For a proof of linearity of Schrödinger flows, given that the property lattice of the corresponding system is $L_{\mathcal{H}}$, see Faure *et al.* (1995). For a proof that the tensor product of Hilbert spaces is appropriate to describe compoundness for systems with as property lattice $L_{\mathcal{H}}$ see Coecke (2000). Conclusively, if the space in which we describe the system is linear, then causal duality forces temporal propagation and compoundness to be described by linear maps. These results essentially use Faure and Frölicher (1993, 1994).

is within environment $\varphi_{\{a,a'\}}$ we condition on the fact that a is obtained—note however, not necessarily in a causal manner, i.e., a does not have to be actual before the measurement. A concrete way to envision this specific situation is in terms of a filter, that whenever the outcome corresponding to a is not obtained the system will be destroyed.[34] Let us denote the corresponding environment as φ_a. Now, since the Sasaki projection $\varphi_a^* : L_{(1)} \to L_{(2)}$ with $t_2 = t_1 + \epsilon$, encodes the behavior of a system under a perfect quantum measurement $\varphi_{\{a,a'\}}$ when the outcome corresponding to a is obtained, it encodes the propagation of properties with respect to φ_a (justifying the notation φ_a^* in perspective of the previous paragraph) and should as such admit a left adjoint expressing (backward) causal assignment, and this is exactly how $(a \stackrel{S}{\to} -) = \varphi_{a,*}(-) : L_{(2)} \to L_{(1)}$ arises in this setting. Sasaki adjunction constitutes as such an incarnation of causal duality. This already 'partly' explains the title of this paper — a more compelling perspective will be discussed in the next section. In particular we will show how causal duality extends to a dynamic logical setting. First we need to introduce causal relations as a dynamic counterpart to the static ordering of properties in the property lattice.

Along the lines of the heuristics behind eq. (3) we can introduce the following two relations, whenever an environment e is specified, *taking a t_1-perspective*:

$$\stackrel{e}{\rightsquigarrow} \subseteq L_1 \times L_2 : a_1 \stackrel{e}{\rightsquigarrow} a_2 \Leftrightarrow \text{"actuality of } a_1 \text{ at } t_1 \text{ implies } \Box\text{-actuality of } a_2 \text{ at } t_2\text{"}; \tag{40}$$

$$\stackrel{e}{\looparrowleft} \subseteq L_1 \times L_2 : a_1 \stackrel{e}{\looparrowleft} a_2 \Leftrightarrow \text{"}\Box\text{-actuality of } a_2 \text{ at } t_2 \text{ implies actuality of } a_1 \text{ at } t_1\text{"}. \tag{41}$$

Now, what do we mean by *taking a t_1-perspective*, and, *□-actuality*? From the perspective at time t_1, i.e., before the interaction of the system and the environment e takes place, there are two modes of envisioning actuality at time t_2, namely i. "a_2 can be actual," the uncertainty being due to the indeterministic nature of the interaction of the system with the environment, and, ii. "a_2 will be actual," definitely. Note that for deterministic transitions these two coincide. Motivated by the modal logic symbolism, we can refer to these two alternatives respectively as *◇-actuality* and *□-actuality*[35] — whenever we mention *◇-actuality* and *□-actuality* we implicitly refer to a t_1-perspective.[36] In general, we clearly have for a fixed property a_2 at t_2 that $\Box -$ actuality $\Longrightarrow \Diamond-$actuality. Formally, this gives us the

[34] See also Piron (1976) on measurements as filters and see Smets (2001) for a recent survey.

[35] For the use of modal-operators in static operational quantum logic, where the operators point out the so-called 'classical limit properties' we refer to Coecke *et al.* (2001a) and Smets (2001, § 10). This however should not be confused with the association to modalities made in this paper.

[36] Note here that the notions *◇-actuality* and *□-actuality* have only significance with respect to a t_1-perspective. In particular, referring to the two modes of envisioning actuality at time t_2 in the t_1-perspective, in a t_2-perspective there is only one since the interaction of the system with the environment did take place.

following (semantical) definitions for $\overset{e}{\rightsquigarrow}$ and $\overset{e}{\looparrowleft}$:

$$a_1 \overset{e}{\rightsquigarrow} a_2 \Longleftrightarrow (\forall p)_1(p \models a_1 \Rightarrow \tilde{e}^*(\{p\}) \models a_2) \tag{42}$$

$$a_1 \overset{e}{\looparrowleft} a_2 \Longleftrightarrow (\forall p)_1(p \models a_1 \Leftarrow \tilde{e}^*(\{p\}) \models a_2) \tag{43}$$

where the index 1 in $(\forall p)_1$ refers to the fact that we quantify over states at time t_1, where $\tilde{e}^*(\{p\})$ denotes the states the system can have after interaction with the environment e and where $\tilde{e}^*(\{p\}) \models a_2$ stands for $\forall q \in \tilde{e}^*(\{p\}) : q \models a_2$. More explicitly referring to eq. (3) we see that as such the relations $\overset{e}{\rightsquigarrow}$ and $\overset{e}{\looparrowleft}$ can be defined in terms of the actuality relation. The major advantage of taking an *a priori* t_1-perspective is that it will allow us to introduce binary connectives that extend this relation 'with the same codomain,' this extension is to be envisioned in the sense that the relation $\leq \subseteq L \times L$ has been extended to an implication—*sensu* eq.(31) and (32)—namely $(- \rightarrow_{\mathrm{DI}(L)} -)$, provided that we considered the disjunctive extension $\mathrm{DI}(L)$ of L and not just L itself. When asking the question whether in some manner the relations $\overset{e}{\rightsquigarrow}$ and $\overset{e}{\looparrowleft}$ indeed extend to connectives it will as such be no surprise that we should again rather consider $\mathrm{DI}(L_i)$ than L_i itself. By 'with the same codomain,' we mean that both will be represented in $\mathrm{DI}(L_i)$—note here indeed that we do not require $L_1 \cong L_2$ and as such also not $\mathrm{DI}(L_1) \cong \mathrm{DI}(L_2)$. (Obviously, all this requires to some extend a pluralistic attitude, we indeed admit that: One could for example find a motivation to consider a t_2-perspective; this then leads us to a bouquet of definable causal relations and corresponding dynamic implications.) Note also that the necessity of having to consider the disjunctive extension already follows from the following observation: given a specified referential frame, one could define an environment *freeze* (with obvious significance), for which it clearly follows that $(- \overset{freeze}{\rightsquigarrow} -) \equiv (- \leq -) \equiv (- \overset{freeze}{\looparrowleft} -)$, i.e., *freeze* provides a *static limit* for the more general *dynamic formalism* that involves explicitation of the environment. As it can be seen in eq.(42) and (43), all these considerations will involve introducing the notions of propagation of propositions and causal assignment for propositions, or equivalently, in terms of the corresponding sets of states that make a proposition true with respect to actuality of one of its members, e.g., $\tilde{e}(\{p\})$. We will do this in the next paragraph. First we take a look on how these relations are realized for the above discussed heuristics for the Sasaki adjunction. Following Coecke *et al.* (2001), we obtain respectively by the definitions of φ_a^* and $\overset{\varphi_a}{\rightsquigarrow}$, and, explicit expression of causal duality, that[37]

$$\varphi_a^*(a_1) \leq a_2 \Longleftrightarrow a_1 \overset{\varphi_a}{\rightsquigarrow} a_2 \Longleftrightarrow a_1 \leq \varphi_{a,*}(a_2), \tag{44}$$

[37] Indeed, $\varphi_a^*(a_1) \leq a_2 \Longrightarrow a_1 \overset{\varphi_a}{\rightsquigarrow} a_2$ follows from the definition of $\leq$, and $\varphi_a^*(a_1) \leq a_2 \Longleftarrow a_1 \overset{\varphi_a}{\rightsquigarrow} a_2$ follows from the fact that $\varphi_a^*(a_1)$ is the strongest property who's actuality is implied by that of a_1 and as such implies any other of that kind.

from which it also follows that $a_1 \overset{\varphi_a}{\rightsquigarrow} \varphi_a^*(a_1)$, that $\varphi_{a,*}(a_2) \overset{\varphi_a}{\rightsquigarrow} a_2$, and in particular, using $\varphi_a^*(1) = a$, that $1 \overset{\varphi_a}{\rightsquigarrow} a$. By the second equivalence in eq. (44) we moreover obtain

Corollary 5.1. $b \vdash (a \overset{S}{\to} c) \Longleftrightarrow b \overset{\varphi_a}{\rightsquigarrow} c.$

The case of the backward relation is less straightforward (and in a sense also less canonical). Indeed, both in the definitions of e^* and e_* we use a forwardly expressed condition in terms of "actuality at t_1 guarantees actuality at t_2," where the definition of $\overset{e}{\hookleftarrow}$ points backwardly. However, we can quite easily prove a similar result as exposed in eq. (44). We first do this for a general environment e.

Proposition 5.2. $a_1 \geq e_*(a_2) \Longleftrightarrow a_1 \overset{e}{\hookleftarrow} a_2$ *and thus* $a_1 \geq \varphi_{a,*}(a_2) \Longleftrightarrow a_1 \overset{\varphi_a}{\hookleftarrow} a_2.$

Proof: $(\Longleftarrow)$: From $p \models e_*(a_2)$, by definition of $e_*(a_2)$ as "guarantees actuality of a_2 at t_2," it follows that $\tilde{e}^*(\{p\}) \models a_2$, so by $a_1 \overset{e}{\hookleftarrow} a_2$ we then obtain $p \models a_1$ and thus $a_1 \geq e_*(a_2)$. $(\Longrightarrow)$: We will first prove that $e_*(a_2) \overset{e}{\hookleftarrow} a_2$, i.e., $(\forall p)_1(p \models e_*(a_2) \Leftarrow \tilde{e}^*(\{p\}) \models a_2)$. Once this is done, $a_1 \overset{e}{\hookleftarrow} a_2$ given that $a_1 \geq e_*(a_2)$ now follows straightforwardly. Since $e_*(a_2)$ is the weakest property that guarantees actuality of a_2 at t_2 we clearly have $p \models e_*(a_2)$ for all states p at t_1 that guarantee actuality of a_2, i.e., $\tilde{e}^*(\{p\}) \models a_2$, so we do have $e_*(a_2) \overset{e}{\hookleftarrow} a_2$, what completes this proof. □

Corollary 5.2. $(a \overset{S}{\to} c) \vdash b \Longleftrightarrow b \overset{\varphi_a}{\hookleftarrow} c.$

Corollary 5.3. $b = (a \overset{S}{\to} c) \Longleftrightarrow b \overset{\varphi_a}{\rightsquigarrow} c \;\&\; b \overset{\varphi_a}{\hookleftarrow} c.$

Note that as a part of the proof of Proposition 3, we obtained $e_*(a_2) \overset{e}{\hookleftarrow} a_2$, and that by Proposition 3 itself we obtain an alternative way of defining causal assignment e_*, namely as:

ii′. 'Causal assignment of properties' $e_* : L_2 \to L_1$ that assigns to any property $a_2 \in L_2$ the strongest property $e_*(a_2) \in L_1$ whose actuality is implied by □-actuality of a_2 at time t_2.

This alternative definition clearly exhibits in a more manifest way the *backwardness* of causal assignment, and consequently, of the action of the Sasaki hook. One easily verifies that *contra* eq. (44) for the case of $\overset{e}{\rightsquigarrow}$ there is no obvious expression of $\overset{e}{\hookleftarrow}$ in terms of e^*. The naive idea one could have to propose $e^*(a_1) \geq a_2$

breaks down on the fact that this would imply $e_* \dashv e^*$ what forces e^* and e_* to be mutually inverse, something that in general (obviously) does not hold. We will now proceed by 'extending' the relations $\stackrel{\varphi_a}{\rightsquigarrow}$ and $\stackrel{\varphi_a}{\hookleftarrow}$ to operations.

6. THE SASAKI HOOK WITHIN DYNAMIC OPERATIONAL QUANTUM LOGIC

First note that for general environments e the relations $\stackrel{e}{\rightsquigarrow} \subseteq L_1 \times L_2$ and $\stackrel{e}{\hookleftarrow} \subseteq L_1 \times L_2$ easily extend to $\mathrm{DI}(L_1) \times \mathrm{DI}(L_2)$ by replacing "($\Box$-)actuality of ..." by "truth with respect to ($\Box$-)actuality of a member of ...," explicitly,

$$A_1 \stackrel{e}{\rightsquigarrow} A_2 \Longleftrightarrow (\forall p)_1 (p \models A_1 \Rightarrow \tilde{e}^*(\{p\}) \models A_2) \tag{45}$$

$$A_1 \stackrel{e}{\hookleftarrow} A_2 \Longleftrightarrow (\forall p)_1 (p \models A_1 \Leftarrow \tilde{e}^*(\{p\}) \models A_2) \tag{46}$$

where $\tilde{e}^*(\{p\}) \models A_2$ now stands for $\forall q \in \tilde{e}^*(\{p\}) : q \models A_2$, i.e. $\forall q \in \tilde{e}^*(\{p\}), \exists a_2 \in A_2 : q \models a_2$, and where one verifies that $\downarrow a_1 \stackrel{e}{\rightsquigarrow} \downarrow a_2 \Longleftrightarrow a_1 \stackrel{e}{\rightsquigarrow} a_2$ and $\downarrow a_1 \stackrel{e}{\hookleftarrow} \downarrow a_2 \Longleftrightarrow a_1 \stackrel{e}{\hookleftarrow} a_2$. Thus we can write elements of the image of $\mathcal{R}$, i.e., those elements in $\mathrm{DI}(L)$ that represent properties, by the properties themselves. In view of eq.(33), (45) and (46) it seems natural to set

$$\mu(A_1 \stackrel{e}{\rightarrow} A_2) := \{p \in \Sigma_1 | p \models A_1 \Rightarrow \tilde{e}^*(\{p\}) \models A_2\} \tag{47}$$

$$\mu(A_1 \stackrel{e}{\leftarrow} A_2) := \{p \in \Sigma_1 | p \models A_1 \Leftarrow \tilde{e}^*(\{p\}) \models A_2\} \tag{48}$$

indeed yielding an extension of the relations since

$$A_1 \stackrel{e}{\rightsquigarrow} A_2 \Longleftrightarrow (A_1 \stackrel{e}{\rightarrow} A_2) = L \quad \text{and} \quad A_1 \stackrel{e}{\hookleftarrow} A_2 \Longleftrightarrow (A_1 \stackrel{e}{\leftarrow} A_2) = L. \tag{49}$$

One verifies that on their turn $(- \stackrel{e}{\rightarrow} -)$ and $(- \stackrel{e}{\leftarrow} -)$ respectively define two tensors $(- \otimes_e -)$ and $(- {}_e\!\otimes -)$ via adjunction (Coecke, 2002b; Coecke *et al.*, 2001b; Smets, 2001), i.e.,

$$(A \otimes_e -) \dashv (A \stackrel{e}{\rightarrow} -) \quad \text{and} \quad (-_e \otimes A) \dashv (- \stackrel{e}{\leftarrow} A). \tag{50}$$

In order to understand the significance of these tensors, first observe that the causal duality derived in the previous section for maps respectively expressing propagation and causation for properties, can also be derived for maps expressing propagation and causation of the propositions in $\mathrm{DI}(L)$, or equivalently, expressing propagation and causation of sets of states in $\mathrm{F}(\Sigma)$. This actually corresponds to forgetting about the existence of $\mathcal{R}$ and applying the construction towards causal duality as if $\mathrm{DI}(L) \cong \mathrm{F}(\Sigma)$ is the lattice of properties of a classical system. The existence of a right adjoint of the map $\hat{e}^* : \mathrm{DI}(L_1) \rightarrow \mathrm{DI}(L_2)$

that assigns to $A_1 \in \mathrm{DI}(L_1)$ the strongest proposition in $\mathrm{DI}(L_2)$ of which truth (i.e., actuality of a member) is implied by that of A_1, or equivalently, of the map $\tilde{e}^* : \mathrm{F}(\Sigma_1) \to \mathrm{F}(\Sigma_2)$ that assigns to $T_1 \in \mathrm{F}(\Sigma_1)$ the collection in $\mathrm{F}(\Sigma_2)$ of obtainable outcome states given that the initial state is in T_1, then imply preservation of respectively $\bigvee_{\mathrm{DI}(L)}$ and $\bigcup$, i.e., disjunction. Complementary, since existence of a right adjoint for propagation of properties encodes preservation of joins for properties, it follows from Coecke and Stubbe (1999) and Coecke (2002a) that $\bigvee_L A = \bigvee_L B \Longrightarrow \bigvee_L \hat{e}^*(A) = \bigvee_L \hat{e}^*(B)$, or, expressed within $\mathrm{DI}(L)$,

$$\mathcal{R}_{\mathrm{DI}(L)}(A) = \mathcal{R}_{\mathrm{DI}(L)}(B) \Longrightarrow \mathcal{R}_{\mathrm{DI}(L)}(\hat{e}^*(A)) = \mathcal{R}_{\mathrm{DI}(L)}(\hat{e}^*(B)). \tag{51}$$

Thus, a shift "from classical to quantum" implies, besides the emergence of the operation $\mathcal{R}$, that classical "preservation of disjunction" becomes a pair consisting of i. preservation of disjunction, and, ii. the continuity-like condition of eq. (51). Thus, coexistence of laws on propagation at the level of L and $\mathrm{DI}(L)$ is not a redundancy.[38] One now verifies that

$$(L \otimes_e -) = \hat{e}^*(-) \quad \text{and} \quad (L_e\!\otimes -) = \hat{e}_*(-), \tag{52}$$

from which follow preservation properties with respect to meet and join, additionally to the ones that follow from the fact that the tensors encode the left-adjoint actions to the hooks.[39]

How does all this apply to the context of quantum measurements, and as such, how does the Sasaki adjunction fits in at this point. First note that we have

$$\hat{\varphi}_a^* : \mathrm{DI}(L_1) \to \mathrm{DI}(L_2) : B \mapsto \bigvee_{\mathrm{DI}(L)} \{\downarrow \varphi_a^*(b) | b \in B\}; \downarrow b \mapsto \downarrow \varphi_a^*(b), \tag{53}$$

i.e., φ_a^* and $\hat{\varphi}_a^*$ act in the same on properties due to the eliminated emergence of disjunction in φ_a. Do we have the same correspondence for the action of $\varphi_{a,*}$ and $\hat{\varphi}_{a,*}$?

Proposition 6.1. *Given $f^* \dashv f_* : L_1 \to L_2$ and $\hat{f}^* \dashv \hat{f}_* : \mathrm{DI}(L_1) \to \mathrm{DI}(L_2)$, then $\downarrow f^*(-) = \hat{f}^*(\downarrow -)$ on L_1 implies $\downarrow f_*(-) = \hat{f}_*(\downarrow -)$ on L_2.*

Proof: For $a, b \in L : \hat{f}^*(\downarrow b) \subseteq \downarrow a \Longleftrightarrow \downarrow \hat{f}^*(b) \subseteq \downarrow a \Longleftrightarrow \hat{f}^*(b) \leq a \Longleftrightarrow b \leq f_*(a)$ so $\hat{f}^*(\downarrow a) = \bigvee_{\mathrm{DI}(L_2)} \{B \in \mathrm{DI}(L_2) | \hat{f}^*(B) \subseteq \downarrow a\} = \bigvee_{\mathrm{DI}(L_2)} \{\downarrow b | b \in L_2, b \leq f_*(a)\} = \bigvee_{\mathrm{DI}(L_2)} \{\downarrow f_*(a)\} = \downarrow f_*(a)$. □

[38] Note here that where L induced a closure on $\mathrm{DI}(L)$, we formally obtain in this case a restriction on the corresponding hom-sets $\mathbf{SL}(\mathrm{DI}(L_1), \mathrm{DI}(L_2))$ in $\mathbf{SL}$, the category of complete lattices and join-preserving maps. As shown in Coecke and Stubbe (1999), the physically admissible transitions constitute a subset of $\mathbf{SL}(\mathrm{DI}(L_1), \mathrm{DI}(L_2))$ for which there exists a 'quantaloid morphism' $\mathrm{R} : \mathbf{SL}(\mathrm{DI}(L_1), \mathrm{DI}(L_2)) \to \mathbf{SL}(L_1, L_2)$.

[39] It turns out that $(- \otimes_e -)$ and $(- \,_e\!\otimes -)$ respectively provide $\mathrm{DI}(L)$ with the structure of a commutative quantale and an in general non-commutative dual quantale (Coecke, 2002b; Coecke *et al.*, 2001b).

Thus, $\hat{\varphi}_{a,*}$ acts on properties as the Sasaki hook does, and it makes therefore sense to set

$$(a \stackrel{S}{\to} -) := \hat{\varphi}_{a,*}(-) : \mathrm{DI}(L) \to \mathrm{DI}(L). \tag{54}$$

What do we obtain in case for eq. (47) and (48) in particular for properties as arguments? Setting ${}^{c}T := \Sigma \setminus T$ we obtain[40]

$$\mu(a_1 \stackrel{\varphi_a}{\to} a_2) = {}^{c}\mu(a_1) \cup \mu(a \stackrel{S}{\to} a_2) \quad \text{and}$$
$$\mu(a_1 \stackrel{\varphi_a}{\leftarrow} a_2) = {}^{c}\mu(a \stackrel{S}{\to} a_2) \cup \mu(a_1). \tag{55}$$

Note here in particular that the Sasaki hook does appear in the expression of both $(a_1 \stackrel{\varphi_a}{\to} a_2)$ and $(a_1 \stackrel{\varphi_a}{\leftarrow} a_2)$, however with a different antecedent than $(a_1 \stackrel{\varphi_a}{\to} a_2)$ and $(a_1 \stackrel{\varphi_a}{\leftarrow} a_2)$. One verifies that using

$$(a_1 \stackrel{\varphi_a}{\to} a_2) = (a_1 \to_L \varphi_{a,*}(a_2)) \quad \text{and} \quad (a_1 \stackrel{\varphi_a}{\leftarrow} a_2) = (\varphi_{a,*}(a_2) \to_L a_1), \tag{56}$$

obtained via adjointness of φ_a^* and $\varphi_{a,*}$ and Proposition 4, and eq. (29) and (30), we obtain

$$(a_1 \stackrel{\varphi_a}{\to} a_2) \;=\; \{b \in L_1 | a_1 \wedge b \le (a \stackrel{S}{\to} a_2)\} = (a_1 \to_L (a \stackrel{S}{\to} a_2))$$
$$\text{and} \quad (a_1 \stackrel{\varphi_a}{\leftarrow} a_2) \;=\; \{b \in L_1 | (a \stackrel{S}{\to} a_2) \wedge b \le a_1\} = ((a \stackrel{S}{\to} a_2) \to_L a_1) \tag{57}$$

i.e., respectively a forward and a backward dynamic φ_a-modification of the static hook $(- \to_L -)$. Similar equations can be obtained for arguments in $\mathrm{DI}(L_1) \times \mathrm{DI}(L_2)$. For the tensors we obtain

$$(A_1 \otimes_{\varphi_a} A_2) = \hat{\varphi}_a^*(A_1 \wedge_{\mathrm{DI}(L)} A_2) \quad \text{and} \quad (A_1 \,{}_{\varphi_a}\!\otimes A_2)$$
$$= (A_1 \wedge_{\mathrm{DI}(L)} (a \stackrel{S}{\to} A_2)). \tag{58}$$

By construction we have the following modified versions of deduction and modus ponens (we express them for properties in analogy to eq. (8) and (9), the significance of the tensors is obvious):

$$b \otimes_{\varphi_a} c \vdash d \Rightarrow c \vdash (b \stackrel{\varphi_a}{\to} d) \quad b \otimes_{\varphi_a} (b \stackrel{\varphi_a}{\to} c) \vdash c \qquad b \,{}_{\varphi_a}\!\otimes c \vdash d$$
$$\Rightarrow b \vdash (d \stackrel{\varphi_a}{\leftarrow} c) \quad (c \stackrel{\varphi_a}{\leftarrow} b)_{\varphi_a} \otimes b \vdash c. \tag{59}$$

We give a global overview of where the Sasaki operations fit in within DOQL—we introduced the notations $\downarrow [L] := \{\downarrow a | a \in L\}$, $(- \stackrel{S}{\leftarrow} a) := (a \stackrel{S}{\to} -)$ and

[40] E.g., via $\mu(a_1 \stackrel{\varphi_a}{\to} a_2) = \{p \in \Sigma_1 \mid p \models a_1 \Rightarrow \varphi_a^*(p) \models a_2\} = \{p \in \Sigma_1 \mid p \models a_1 \Rightarrow p \models \varphi_{*,a}(a_2)\}$ $= {}^{c}\mu(a_1) \cup \mu(a \stackrel{S}{\to} a_2)$.

$(- \leftarrow_X \ \cdot) := (\cdot \rightarrow_X -)$ for $X \in \{L, \mathrm{DI}(L)\}$.[41]

Statical (freezed dynamical)	φ_a-induced forward **dynamical**	φ_a-induced backward **dynamical**
$id_L : L \to L$	$\varphi_a^*(-) : L_1 \to L_2$	$\varphi_{a,*}(-) = (- \stackrel{S}{\leftarrow} a) : L_2 \to L_1$
$id_{\mathrm{DI}(L)} : \mathrm{DI}(L) \to \mathrm{DI}(L)$	$\hat{\varphi}_a^*(-) = \downarrow \varphi_a^*(-) on \downarrow [L_1]$	$\hat{\varphi}_{a,*}(-) = \downarrow (- \stackrel{S}{\leftarrow} a)\ on \downarrow [L_2]$
$(- \rightarrow_L -) : L \times L \to \mathrm{DI}(L)$	$(- \stackrel{\varphi_a}{\rightarrow} -) = (- \rightarrow_L (a \stackrel{S}{\rightarrow} -))$	$(- \stackrel{\varphi_a}{\leftarrow} -) = (- \leftarrow_L (- \stackrel{S}{\leftarrow} a))$
$(- \rightarrow_{\mathrm{DI}(L)} -)$ on $\mathrm{DI}(L)$	$(- \stackrel{\varphi_a}{\rightarrow} -) = (- \rightarrow_{\mathrm{DI}(L)} (a \stackrel{S}{\rightarrow} -))$	$(- \stackrel{\varphi_a}{\leftarrow} -) = (- \leftarrow_{\mathrm{DI}(L)} (- \stackrel{S}{\leftarrow} a))$
$(- \wedge_{\mathrm{DI}(L)} -)$ on $\mathrm{DI}(L)$	$(- \otimes_{\varphi_a} -) = \hat{\varphi}_a^*(- \wedge_{\mathrm{DI}(L)} -)$	$(-_{\varphi_a}\otimes -) = (- \wedge_{\mathrm{DI}(L)} (- \stackrel{S}{\leftarrow} a))$

Using eq. (56) we can actually formally recover the Sasaki hook and projections as

$$(1 \stackrel{\varphi_a}{\rightarrow} -) = (1 \rightarrow_L (a \stackrel{S}{\rightarrow} -)) = (a \stackrel{S}{\rightarrow} -)$$
$$(1 \otimes_{\varphi_a} -) = \varphi_a^*(1 \wedge -) = \varphi_a^*(-). \tag{60}$$

In the static case both of these become the identity, i.e., $(1 \rightarrow_L -) = (1 \wedge -) = id_L$, this giving the Sasaki hook and projectors a formal interpretation as dynamic modifications of the identity. Let us conclude this section with the following identities

$$((a \stackrel{S}{\rightarrow} b) \stackrel{\varphi_a}{\rightarrow} b) = L \qquad ((a \stackrel{S}{\rightarrow} b) \stackrel{\varphi_a}{\leftarrow} b) = L \tag{61}$$

or, differently put

$$(a \stackrel{S}{\rightarrow} b) \stackrel{\varphi_a}{\rightsquigarrow} b \qquad (a \stackrel{S}{\rightarrow} b) \stackrel{\varphi_a}{\hookleftarrow} b \tag{62}$$

what formally encodes our interpretation of what the Sasaki hook does.

7. QUANTUM LOGIC RESEARCH? HOW TO CONVERT?

As mentioned above, Coecke (2002a), the traditional domain of study of quantum-like lattices should now be envisioned as a study of the range of the operation $\mathcal{R}$, with corresponding heuristics. This obviously sheds a different light on the significance of, for example, orthomodularity. Weak modularity actually does not come in at all in the disjunctive extension construction—we recall from Coecke (2002a) that orthocomplementation comes in the sense that operational resolution has an involutive square-root named the 'operational complementation.' However, since orthomodularity is equivalent with the Sasaki adjunction and this Sasaki adjunction in its turn represents causal duality, we have provided a new dynamic interpretation of the axiom of orthomodularity. But, we can push this further. The

[41] We note that there are some subtilities which we did not mention for sake of transparency of the argument, in particular with respect to $\varphi_a^*(b) = 0$ where we have two options: i. introduce a kernel of inadmissible initial states, i.e., consider $\tilde{\varphi}_a^* : \mathrm{F}(\Sigma \setminus K) \to \mathrm{F}(\Sigma)$, or, consider upper pointed extensions *sensu* (Coecke *et al.*, 2001; Sourbron, 2001). Note that similar considerations can be made for environments $\varphi_{\{a,a'\}}$ instead of φ_a though then we truly obtain two levels, one for $\varphi_{\{a,a'\}}^*$ and $\varphi_{\{a,a'\}*}$ and one for $\hat{\varphi}_{\{a,a'\}}^*$ and $\hat{\varphi}_{\{a,a'\}*}$.

essential conclusion that comes out of our analysis is that in traditional quantum logic "too much was encoded in too little": properties where identified with propositions, temporal phenomena like 'change of state in a measurement' where statically encoded, the distinction between the structure of properties and event structures was in many occasions mixed up, etc. What about event structures? Since we claimed that there was no conflict between an endo- and an exo-perspective, where could they fit in in our setting? They clearly should lie at the base of structuring the collection of environments.[42] From a logical perspective, this implies a two-dimensional situation: a structure of environments that interacts with a structure of propositions on a system. Of particular interest would then be the case where we restrict to environments $\{\varphi_a | a \in L\}$, as we essentially did in this paper. This would mean that the environments are structured in the same way as the image of $\mathcal{R}$, order-isomorphic to the properties L, so this realizes a structure in which the lattice of closed subspaces of a Hilbert space appears both as the properties and as labels encoding (physical) environments, where actually we rather think about the latter as being the projectors in standard quantum theory. In that sense the situation of "too much being encoded in too little" gets explicitly unraveled, and motivates that it truly seems to make sense to distinguish between closed subspaces and projectors at an abstract structural level although they are in bijective correspondence, the first having an ontological connotation, the second an empirical. This then leads to the perspective that the transition from either classical or constructive/intuitionistic logic to quantum logic entails besides the introduction of an additional unary connective operational resolution $\mathcal{R} : \mathrm{DI}(L) \to \mathrm{DI}(L)$ the shift from a binary connective implication to a ternary connective

$$(- \stackrel{-}{\to} -) : \mathrm{DI}(L) \times \mathcal{R}[\mathrm{DI}(L)] \times \mathrm{DI}(L) \to \mathrm{DI}(L). \tag{63}$$

where two of the arguments refer to qualities of the system and the third, the new one, to an obtained outcome (in a measurement). A better way of putting things would be

$$(- \stackrel{-}{\to} -) : \mathrm{DI}(L) \times \mathbb{P}(L) \times \mathrm{DI}(L) \to \mathrm{DI}(L). \tag{64}$$

where $\mathbb{P}(L)$ are the projectors (on corresponding closed subspaces) what for general orthomodular lattices ends up being the Baer *-semigroup of projectors in the Foulis (1960) sense. In view of Piron's (1964) theorem it then follows that this situation fully covers 'pointless quantum theory'—sensu pointless topology in terms

[42] Some very general attempts in this direction were initiated in Amira *et al.* (1998). We also mention that in the Foulis–Randall perspective there are some recent attempts by Greechie and Gudder (2001) to build so-called sequential effect algebras that provide a dynamic structure for discussing empirical events. Along the lines of Section 2 one could say that sequential effect algebras aim at characterizing the structure of consecutive measurements as a reflection of the system's behavior there where the aim in Amira *et al.* (1998) is essentially to obtain a theory on the system's behavior itself, incorporating the interaction with its environment possibly including a measurement setup.

of locales, i.e., complete Heyting algebras—since we drop the two point-related axioms atomisticity and covering law in order to have a (complete) orthomodular lattice. Another extrapolation of the setting presented in this paper consists of rather than defining dynamic operations (labeled by environments) on the static propositions DI(L), we start of from 'dynamic propositions', i.e., propositions on 'propagation of (actual) properties' rather than on actuality itself and here inspiration can be found in research within the domain of for example computational process semantics (Milner, 1999), action logic (Baltag, 1999), etc. Obviously, much is still to be done in that direction.

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